AN INTRODUCTION TO GAME THEORY

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  • What is Game Theory?
  • Classification

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      • Nash equilibrium
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AN OVERVIEW
What is Game Theory?

• Goals
  It aims to help us understand situations in which decision-makers (players) interact: Interactive Decision Theory

• Origins

• Application Areas
  Economics, political science, psychology, computer science

Assumption: the players are rational (have a clear relation of preferences over the outcomes\(^1\)) and intelligent (are able to act in a rational way)

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1 Axioms of "rationality" (Von Neumann–Morgenstern utility theorem, 1947)
Classification

• Non-Cooperative and Cooperative Game
  - Non Cooperative: binding agreements are not allowed
  - Cooperative: binding agreement are allowed

• Games with Perfect and Imperfect Information
  - Games with Perfect Information: the players are fully informed about the possible moves of the others players
  - Games with Imperfect Information: the players have only partial information about the possible moves of the others players

• Games in Extensive, Strategic and Characteristic form
  - Extensive form: detailed description of the game (before 1944)
  - Strategic form: game in normal form; Von Neumann-Morgenstern (1944)
  - Characteristic form: for cooperative games only
Strategic form games

- Extensive form games
  - Characteristic form games
    - Non cooperative games
      - Rationalizability
        - Correlated equilibrium
          - (Evolutionary equilibrium)
        - Sequential and trembling hand perfect equilibrium
    - Perfect information
    - Sequential and trembling hand perfect equilibrium

- Cooperative games
  - Core
    - Bargaining games
      - Repeated games
        - Imperfect information
          - Core

- Coalitional games
  - Imperfect information
    - Bayesian games
      - Nash equilibrium
      - Extensive form games

- Nash equilibrium
NON-COOPERATIVE GAMES

Strategic Games
Definition of Strategic Game

«A model of interaction among decision makers. Each player chooses his ‘plane of action’ once and for all and the choices are made simultaneously.».

- a finite set \( N \) (players)
- for each player \( i \in N \)
  - a nonempty set \( S_i = \{s_i^1, s_i^2, \ldots\} \) (set of strategies available to \( i \))
  - a preference relation \( \succeq_i \) on \( S = \times_{j \in N} S_j \) (set of outcomes or profiles)
    - a profile \( s \) is a \( N \)-pla of strategies \( s = (s_j^{k(j)})_{j \in N} \)

A preference relation \( \succeq_i \) is a function \( u_i : S \rightarrow \mathbb{R} \) (payoff)

\[ s_1 \succeq_i s_2 \iff u_i(s_1) \geq u_i(s_2) \]
Games in strategic form: examples

Prisoner’s Dilemma

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Battle of sexes

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Head and Tail

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Pure Coordination

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Non-cooperative strategic games:
- **one-shot** games
- **repeated** games *(the strategic model is appropriate only if there are no strategic ties among the repetitions)*
Some notation and definitions

• Some notation
  - If \( s = (s_i)_{i \in N} \) is a strategy profile, then \( s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N) \)
  - \( (s_i, s_{-i}) = s \)

• Definitions
  - \( s_i \) is a **best response** to \( s_{-i} \) if
    \[
    u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \text{for every strategy } s_i' \text{ available to } i
    \]
  - \( s_i \) is a **unique best response** to \( s_{-i} \) if
    \[
    u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{for every } s_i' \neq s_i
    \]
STRATEGIC GAMES

Solution Concepts
Solution concepts

• In Game Theory (multiple agents or players) the a ‘best strategy’ for a player depends on others’ choices.

  Solution concepts = ‘subsets of outcomes (profiles) which are in some sense preferable’.

• Some solution concepts (non-cooperative strategic games):
  - Pareto optimality
  - Dominant Strategy equilibrium
  - Nash equilibrium
  - Iterated elimination of Strictly Dominated Actions (Rationalizablility)
  - Mixed strategies Nash equilibrium
  - Correlated equilibrium

  non deterministic player’s strategies
Pareto optimality

• The strategy profile \( s \) **pareto dominates** a strategy profile \( s' \) if
  - no agent gets a worse payoff with \( s \) than with \( s' \)
    i.e. \( u_i(s) \geq u_i(s') \) for all \( i \)
  - at least one agent gets a better payoff with \( s \) than with \( s' \)
    i.e. \( u_i(s) > u_i(s') \) for at least one \( i \)

• A strategy profile \( s \) is **Pareto optimal** or **strongly Pareto efficient** if there is no strategy \( s' \) that Pareto dominates \( s \)
  - every game has at least one Pareto optimal profile
  - there is always at least one Pareto optimal profile in which the strategies are pure
Example

The Prisoner’s Dilemma

- (NC,NC) is Pareto optimal
  - no profile gives both players a higher payoff

- (NC,C) is Pareto optimal
  - no profile gives player I a higher payoff (or at least equal)

- (C,NC) is Pareto optimal
- (C,C) is Pareto dominated by (NC,NC)

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Dominant strategy

Definition

Let \( S_i = \{s^1_i, s^2_i, \ldots\} \) the set of all the strategies available to agent \( i \)

- The strategy \( s^k_i \) **strongly dominates** \( s^h_i \) for player \( i \) if
  \[
  u_i(s^k_i, s_{-i}) > u_i(s^h_i, s_{-i}) \quad \forall s_{-i}
  \]

- The strategy \( s^k_i \) **weakly dominates** \( s^h_i \)
  \[
  u_i(s^k_i, s_{-i}) \geq u_i(s^h_i, s_{-i}) \quad \forall s_{-i}
  \]
  \[
  u_i(s^k_i, s_{-i}) > u_i(s^h_i, s_{-i}) \quad \text{for some } s_{-i}
  \]

- \( s^k_i \) is a (strongly, weakly) dominant strategy if (strongly, weakly) dominates every \( s^h_i \in S_i \)
Dominant strategy equilibrium

- A **dominant strategy equilibrium** is a profile \( S = (s_1, \ldots, s_N) \) such that \( s_i \) is dominant for the player \( i \).
- Each player \( i \) do best by using \( s_i \) rather than a different strategy, regardless of what strategy the other players use.

**Example** (The Prisoner’s Dilemma)

- there is one dominant strategy equilibrium: (C,C)
  - both player defect
  - it is not *Pareto optimal*

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• It is a **stronger concept than the Nash equilibrium**
Nash equilibrium

The most important solution concept for non-cooperative games

**Definition (pure strategy Nash equilibrium)**

A strategy profile \( s^* = (s_1^*, ..., s_N^*) \) is a Nash equilibrium if for every player \( i \) if

\[
    u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \text{for every } s_i \in S_i
\]

i.e. for every player \( i \) \( s_i^* \) is the best response to \( s_{-i}^* \) / no player can yield an higher payoff by unilaterally changing his strategy.

- Interpretation: steady state

- **Dominant Strategy equilibrium** ↔ Nash equilibrium
Examples (N = 2)

Prisoner’s Dilemma

OSS: A Nash equilibrium is inefficient when is not pareto optimal.

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Generalization and Refinements

• Generalization: *Mixed strategies Nash equilibrium*
• A further generalization of the Nash equilibrium concept is the rationalizability

Oddities in the Nash equilibrium:

- inefficiency (Prisoner’s dilemma)
- non-uniqueness (Battle of sexes, Pure coordination)
- non-existence (Head and Tail)

• In order to avoid the non-existence and multiple Nash equilibria:
  1. *Correlated equilibrium*
  2. *Perfect subgame equilibrium*
  3. *Trembling hand perfect equilibrium*

All failed w.r.t. uniqueness and efficiency need to account for cooperation (Cooperative games)
Mixed strategies

- Attempt: to generalize the Nash equilibrium concept (pure strategy)
- Probabilistic approach: we each player choose a probability distribution over his set of strategies (independently) instead of choosing a single deterministic strategy

**Definition (Mixed strategy)**
A mixed strategy $\alpha_i$ for player $i$ is a probability distribution over his set of strategies (actions)

- Pure strategy profile: $s = (s_1, s_2, \ldots, s_N)$
- Mixed strategy profile: $\alpha = (\alpha_i)_{i \in N}$ $\alpha \in \Delta(S)$

Given $\alpha$ (p.d. over deterministic outcomes), the expected payoff of player $i$ is a function $U_i : \times j \in N \Delta(A_j) \rightarrow \mathbb{R}$ defined as

$$U_i(\alpha) = \sum_{s \in S} (\prod_{j \in N} \alpha_j(s_j))u_i(s)$$

i.e. the expected value of $u_i : \times j \in N S_j \rightarrow \mathbb{R}$ induced by $\alpha$
Mixed strategy game

Definition
Given $\alpha$ (p.d. over deterministic outcomes), the expected payoff of player $i$ is a function $U_i : \times_{j \in N} \Delta(A_j) \rightarrow \mathbb{R}$ defined as

$$U_i(\alpha) = \sum_{s \in S} \left( \prod_{j \in N} \alpha_j(s_j) \right) u_i(s)$$

i.e. the expected value of $u_i : \times_{j \in N} S_j \rightarrow \mathbb{R}$ induced by $\alpha$

- The strategic game $\langle N, (\Delta(S_i)), (U_i) \rangle$ is the mixed extension of the strategic game $\langle N, (S_i), (u_i) \rangle$

A mixed strategies Nash equilibrium of a strategic game is a Nash equilibrium of the mixed extension
**Mixed strategy Nash equilibrium**

**Definition (Mixed strategies equilibrium)**
A mixed strategy profile is a **mixed strategy Nash equilibrium** if

\[
U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_{-i}) \quad \forall \alpha_i, \forall i \in N \quad (\alpha^*_i \in B_i(\alpha^*_{-i}) \quad \forall i \in N)
\]

**Properties**
- The set of pure strategy equilibria is a *subset* of the set of the mixed strategy equilibria
- Every finite strategic game has a mixed strategy Nash equilibrium (it solves the non-existence problem)
Example (Haid and Tail)

- No Nash equilibrium (in pure strategies)
- Unique mixed strategy Nash equilibrium: \(((1/2,1/2),(1/2,1/2))\)

\[ \alpha = (\alpha_1, \alpha_2) = ((p, 1-p), (q, 1-q)) \]

Player 1’s best expected payoff (best response):

\[ U_1(\alpha/\text{Head}) = q \cdot 1 + (1-q) \cdot (-1) = 2q - 1 \]
\[ U_1(\alpha/\text{Tail}) = q \cdot (-1) + (1-q) \cdot 1 = 1 - 2q \]

\[ q < 1/2 \quad \rightarrow \quad U_1((0,1), \alpha_2) \geq U_1(\alpha_1, \alpha_2) \quad \forall \alpha_1 \]
\[ q > 1/2 \quad \rightarrow \quad U_1((1,0), \alpha_2) \geq U_1(\alpha_1, \alpha_2) \quad \forall \alpha_1 \quad \leftrightarrow \quad B_1(\alpha_2) = \begin{cases} 
(0,1) & q < 1/2 \\
(p, 1-p) & q > 1/2 \\
(1,0) & q > 1/2 
\end{cases} \]

Player 2’s best expected payoff (best response):

\[ B_2(\alpha_1) = \begin{cases} 
(1,0) & p < 1/2 \\
(q, 1-q) & p = 1/2 \\
(0,1) & p > 1/2 
\end{cases} \]
Example (Haid and Tail)

The set of mixed strategy Nash equilibria of the game corresponds to the set of *intersections of the best response function*,

\[ \alpha^* = (B_1(\alpha^*_2), B_2(\alpha^*_1)) \]

\[ \alpha^* = \left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right) \]

The game has a unique Nash equilibrium in mixed strategies.
Example (BoS)

- Two Nash equilibria (in pure strategies)
- Tree Nash equilibria in mixed strategies

\[ \alpha = (\alpha_1, \alpha_2) = ((p, 1-p), (q, 1-q)) \]

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**Player 1’s best response function:**

\[
U_1(\alpha/\text{football}) = 2 \cdot q + 0 \cdot (1 - q) = 2q \\
U_1(\alpha/\text{opera}) = 0 \cdot q + 1 \cdot (1 - q) = 1 - q
\]

\[ \rightarrow \quad B_1(\alpha_2) = \begin{cases} 
(0,1) & q < \frac{1}{3} \\
(p,1-p) & q = \frac{1}{3} \\
(1,0) & q > \frac{1}{3}
\end{cases} \]

**Player 2’s best response function:**

\[ B_2(\alpha_1) = \begin{cases} 
(0,1) & p < \frac{2}{3} \\
(q,1-q) & p = \frac{2}{3} \\
(1,0) & p > \frac{2}{3}
\end{cases} \]
There are three intersection points of the players’ best response functions.

\[
\alpha^* = \left( (0, 1), (0, 1) \right)
\alpha^* = \left( (1, 0), (1, 0) \right)
\alpha^* = \left( \left( \frac{2}{3}, \frac{1}{3} \right), \left( \frac{1}{3}, \frac{2}{3} \right) \right)
\]

are the Nash equilibria (opera, opera) and (football, football) in pure strategies.

mixed strategy Nash equilibrium (each of the 4 deterministic outcomes occurs with positive probability)

OSS: The mixed Nash equilibrium is pareto dominated by the two pure Nash equilibria.
Example (BoS)

There are three intersection points of the players’ best response functions.

\[ \alpha^* = ((0,1), (0,1)) \]
\[ \alpha^* = ((1,0), (1,0)) \]

are the Nash equilibria

(opera, opera) and

(football, football) in pure strategies

\[ \alpha^* = ((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})) \]

mixed strategy Nash equilibrium (each of the 4 deterministic outcomes occurs with positive probability)
Rationalizability

Assumption: each player knows that the other players are intelligent and rational

- A strategy \( S \) is a rationalizable equilibrium if an infinite sequence of reasoning (consistent beliefs) results in the players playing \( S \)

- How to find rationalizable strategies?
  
  to look for non-rationalizable actions and eliminate them

**Def**: an action of player \( i \) is a never-best response if it is not a best response to any belief of player \( i \)

Never-best response \( \rightarrow \) non rationalizable (see the Prisoner’s dilemma)

**Def**: the strategy \( s_i \in S_i \) of player \( i \) is strictly dominated if there exists a mixed strategy \( \alpha_i \in \Delta(S_i) \) of player \( i \) that strictly dominates, i.e.

\[
U_i(\alpha_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i} \quad s_{-i} \in S_{-i}
\]

\( \rightarrow \) A strictly dominated strategy is a never best response

\( S_i \): never best response \( \iff \) strictly dominated
Iterated elimination of Strictly Dominated Actions

1. Eliminate strictly dominated actions from the game because no rational player plays such actions;
2. Even more actions can be strictly dominated within the remaining game; so eliminate them;
3. Further actions can be eliminated since each player is rational, believes that the other players are rational, and believes that the other players believe that the other players are rational……
4. For a finite game, the process of successive eliminations stop at some point; ……obtaining the set of all rationalizable strategies.

- Nash equilibrium  $\iff$  Rationalizable equilibrium

Oss: the rationalizability concept looks at the game from the point of view of a single player
Examples

- **Head and Tail**
  
  No elimination is possible; all the pure strategies in this game are rationalizable.

- **The same happens in any coordination game** (players choose corresponding strategies).
  Es: **Pure coordination game**

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- **Prisoner’s dilemma**

- **Typical example**
Games with communication

- To solve «inefficiency» and «non-uniqueness» of the Nash equilibrium
- Communication among players

**Communication ➔ Cooperation**

- The introduction of communication among players can lead to a Self-enforcing equilibrium (without binding agreement)

*Definition (Generalized strategy)*

A **correlated strategy** or **jointly randomized strategy** for a set of players $C \subseteq N$ is any probability distribution $\alpha$ over the set of possible combinations of pure strategies these players can choose, i.e.

$$\alpha \in \Delta(S_c) = \Delta(\times_{i \in C}(S_i))$$

**Correlated strategy profile vs Mixed strategy profile**

$$\alpha \in \Delta(\times(S_i))$$

$$\alpha \in \times\Delta(S_i)$$

In a correlated strategy the mixed strategies can be correlated
Correlated strategies and equilibrium

A correlated strategy $\alpha$ can be implemented by the players through a mediator which recommends randomly a profile of pure strategies according to $\alpha$.

**Correlated equilibrium** (Aumann, 1974)

«Any correlated strategies for the players which could be self-enforceingly implemented with the help of a mediator who makes non binding recommendations to each player»

- Refinement of the mixed Nash equilibrium
- Includes communication among players (public signal/recommended strategy)
Correlated equilibrium

**Definition**
The expected payoff to player $i$ when a correlated strategy $\alpha \in \Delta(S)$ is implemented is $U_i(\alpha) = \sum_{s \in S} \alpha(s)u_i(s)$

**Mediator suggestion:** $\alpha^* \in \Delta(S)$
- $\delta_i : S_i \rightarrow S_i$, for each player $i$ ( $\delta_i(s_i) = s_i$ means that player $i$ obeys the mediator )

**Definition (Correlated equilibrium)**
The correlated strategy $\alpha^*$ induce an equilibrium for all players to obey the mediator recommendation if and only if

$$U_i(\alpha) = \sum_{s \in S} \alpha(s, s_{-i})u_i(s, s_{-i}) \geq \sum_{s \in S} \alpha(s, s_{-i})u_i(\delta_i(s_i), s_{-i})$$

$\forall i \in N, \forall \delta_i : S_i \rightarrow S_i$
An example

Payoff allocation of pure Nash equilibria: (5,1), (1,5)
mixed Nash equilibrium (2.5,2.5)

Drawback: ‘non-uniqueness’ and ‘inefficiency’

- A better outcome than (2.5, 2.5) can be obtained through correlated strategies
  - es: \( \alpha(x_1, x_2) = \alpha(y_1, y_2) = \frac{1}{2}; \quad \alpha(x_1, y_2) = \alpha(y_1, x_2) = 0 \)
  - is a self-enforcing plan with expected payoff (3,3)

- es: \( \alpha(x_1, x_2) = \alpha(y_1, x_2) = \alpha(y_1, y_2) = \frac{1}{3}; \quad \alpha(x_1, y_2) = 0 \)
  - is a self-enforcing plan with expected payoff \( 3 + 1/3, 3 + 1/3 \)
Properties of Correlated equilibria

- The set of correlated equilibria contains the set of mixed strategies Nash equilibria
- The set of correlated equilibria includes outcomes which are Pareto efficient (not Pareto dominated by the pure Nash equilibria)
- Finding correlated equilibria is computationally less expensive than searching for Nash equilibria (LP problem)
Linear programming problem (LPP)

- The set of correlated equilibria is a *compact* and *convex* set.
- Finding the **correlated equilibrium that maximize the sum of the player’s expected payoff** is equivalent to solve the following LPP:

\[
\max_{i \in N} \sum_{i \in N} U_i(\alpha) \\
\sum_{s_{-i} \in S_{-i}} \alpha(s)[u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i})] \geq 0 \quad \forall i \in N, \forall s_i \in S_i, \forall s'_i \in S_i \\
\alpha(s) \geq 0 \quad \forall s \in S \\
\sum_{s \in S} \alpha(s) = 1.
\]

By solving the linear problem in the previous example, among all the correlated equilibria \( \alpha = \left( \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3} \right) \) is the ‘best’ one.
STRATEGIC GAMES
Games with Imperfect Information
Bayesian Games: an example (1)

Bayesian Games = Games with Imperfect Information in strategic form

**Example** (Variant of BoS with imperfect information)

- two **states** with different Player’s preferences;
- from player 1’s point of view Player 2 has two **types**;
- Player 1 has **beliefs** about the type of Player 2 (coming from experience or updated as the play takes place): $\frac{1}{2}$ and $\frac{1}{2}$
Bayesian Games: an example (2)

- Expected payoffs of Player 1 for the possible pairs of strategies of the two types of Player 2

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*Pure strategy Nash equilibrium* = triple of strategies (one for P1 and one for each type of P2) with the property that

- the strategy of P1 is optimal, given the actions of the two types of P2 (and P1’s belief about the state)
- the action of each type of P2 is optimal, given the action of P1

(B,(B,S)) is a Nash equilibrium

The *types* must be treated as separate *players!*
Bayesian Games

A Bayesian game consists of:

- a set of **players** \( N \)
- a set of **states** \( \omega \in \Omega \)
- a set of **strategies** \( S_i \) for each player \( i \)
- a finite set \( T_i \) of **types** of player \( i \) and a function \( \tau_i : \Omega \rightarrow t_i \) which assigns a type to any state for player \( i \)
- a probability measure \( p_i \) on \( \Omega \) for each player \( i \) (the **prior belief** of \( i \))
- Bernoulli payoffs \( u_i : S \times \Omega \rightarrow \mathbb{R} \) for each player \( i \)

**Definition**

A **Nash equilibrium of a Bayesian Game** is a Nash equilibrium of the strategic game defined as follows

- the set of players \( (i, t_i), \quad i \in N, \ t_i \in T_i \)
- the set of strategies \( S_{(i, t_i)} \) for each player \( (i, t_i) \), \( S = \times S_{(i, t_i)} \)
- the Bernoulli payoffs \( u_{(i, t_i)} : S \rightarrow \mathbb{R} \) for each player \( (i, t_i) \) is the expected payoff of type \( t_i \) of player \( i \)
Bayesian Games

A Bayesian game consists of:

- a set of players \( N \)
- a set of states \( \omega \in \Omega \)
- a set of strategies \( S_i \) for each player \( i \)
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Definition

A Nash equilibrium of a Bayesian Game is a Nash equilibrium of the strategic game defined as follows

- the set of players \( (i, t_i), \ i \in N, t_i \in T_i \)
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